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NONLINEAR DISTURBANCE OBSERVER FOR A CLASS OF NONLINEAR SYSTEMS

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ABSTRACT

A disturbance observer achieves on-line estimation of unknown disturbances acting on a dynamic system based on the input and output information. This paper proposes a nonlinear disturbance observer for a class of nonlinear systems. Arbitrarily close estimate of the unknown disturbance can be obtained if the disturbance is bounded with bounded first-order time derivative, and the system nonlinearity satisfies a Lipschitz condition. A unique feature of the proposed nonlinear disturbance observer is that it allows the Lipschitz constant to be arbitrarily large.

KEYWORDS: Disturbance Observer, Disturbance Estimation, Robust Observer, Nonlinear System, Lipschitz Condition

1. INTRODUCTION

In many engineering and science problems, one is concerned with the estimation of disturbances (or unknown inputs) acting on a dynamic system [1], [2], [3]. In control engineering, *disturbance observer* and *unknown input observer* are the most popular approaches to on-line disturbance estimation. The Disturbance Observer (DOB) [4] and [5] approach refers to using direct system inverse to estimate the input disturbance. In order to avoid a non-proper system inverse, a low-pass Q-filter is cascaded with the system inverse, where the Q-filter is chosen to achieve accurate disturbance estimation and to avoid amplification of measurement noise. Different design considerations and design skills have been proposed in the literature [6] - [11]. Disturbance rejection control based on DOB has been proved to be very effective especially in motion control applications.

The second approach to disturbance estimation is the unknown input observer. In recent literatures, the most popular unknown input observer is the joint state and disturbance observer, where an observer is constructed to simultaneously estimate both the unknown state and the unknown disturbance. The designs in [12] - [14] are applicable under the assumption that the unknown disturbance is constant or slowly varying. The extended state observer design [15] - [17] basically follows the same assumption If knowledge of a disturbance model is known a priori, Luenberger type jointed state and disturbance observer can be constructed in [18] - [21] When no disturbance model is available, the designs in [22], [23] are applicable.

Studies on the disturbance observer designs in linear systems are more common than nonlinear ones in the literature. The work in [22] and [24] simultaneous estimate the unknown disturbance and unknown state for a class of nonlinear systems. Their solution relies on solving an Linear Matrix Inequality (LMI), but the LMI admits a positive definite solution only if the Lipschitz constant of the nonlinearity is small [24]. Further, they constraint the relative degree from the nonlinearity to the system output to be one [22] There has been study of the disturbance observer for a particular class of nonlinear system, that is, the robot systems [25] - [27]. In these literatures, the robot disturbance observer must assume that all the state variables (angles and velocities of all joints) are available for measurement. While the disturbance observer in this paper requires only information of system outputs (for example, angles of certain joints), and this requirement is much more relaxed than the robot disturbance observer designs.

The disturbance observer design problem presented in the paper is more difficult than the state observer design problem for nonlinear systems. For the state estimation problem for nonlinear systems, one may see the interesting work in [28], [29], which shows that a nonlinear state observer exists as long as the distance to unobservability is larger than the Lipschitz constant. For the disturbance observer problem treated in this paper, one must first design a robust nonlinear observer that ensure small estimation error in front of unknown disturbances (Theorem 3 below). Then, one needs to prove that in the state estimation error dynamics, small state estimation error implies small input ex citation. After establishing this property can one prove that the disturbance estimation error is small. The disturbance estimation problem is therefore non trivial, and is dealt with by Lemma 4 and Theorem 5 in this paper.

The remainder of this paper is arranged as follows. Section 2 presents the new nonlinear disturbance observer design for systems whose nonlinearity enters the system equation in the same direction as the disturbance. Section 3 further extends the nonlinear observer so that the nonlinearity and disturbance enter the system equation in different directions. Finally, Section 4 gives the conclusions.

2. NONLINEAR DISTURBANCE OBSERVER

Consider the disturbance estimation problem for a nonlinear system

$$\begin{aligned} \dot{x} &= Ax + Bu + G(f(x) + d), \\ y &= Cx. \end{aligned} \quad \textbf{(1)}$$

Where $x \in R^n$ is the unknown system state, $u \in R^m$ the control input, $y \in R^p$ the measured system output, $d \in R^q$ the unknown disturbance satisfying

$$\|d\| \le D_0, \ \|\dot{d}\| \le D_1, \quad \textbf{(2)}$$

For two positive constants D_0 and D_1 , and $f(x) \in R^q$ is a nonlinearity satisfying the Lipschitz condition:

$$\|f(x) - f(\hat{x})\| \le \gamma \|x - \hat{x}\|, \quad \textbf{(3)}$$

For some Lipschitz constant $\gamma > 0$. Note thaFt in this paper the Lipschitz constant γ can be arbitrarily large while previous papers all constrain γ to be small. *C* and *G* are full rank. It is assumed that

$$\dim(y) \ge \dim(d), \quad \textbf{(4)}$$

Where *dim* stands for dimension. Without loss of generality it is assumed in the sequel that *dim*(*y) =dim* (*d*); the case *dim*(*y*)>*dim* (*d*) can be treated by artificially expanding the dimension of *d*. Further, it is assumed that (*A*, *C*) is observable, (*A*, *G*) controllable, and the square system (*A*, *G*, *C*) has only stable zeros (minimum-phase).

To solve the disturbance estimation problem for the nonlinear system (1), one constructs a nonlinear joint state and disturbance observer as follows.

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) + Gf(\hat{x}), \quad \textbf{(5)}$$

$$\hat{d} = \sqrt{\pi}W(y - C\hat{x}), \quad \textbf{(6)}$$

Where $\hat{d}$ is a disturbance estimate for the unknown disturbance *d*, $W \in R^{p\times p}$ is an unitary matrix to be specified, the observer feedback gain $L \in R^{n\times p}$ is designed as in the LTR observer [30] and [31],

$$L = QC^T, \quad \textbf{(7)}$$

$$Q(A+\alpha I)^T + (A+\alpha I)Q - QC^TCQ + \pi GG^T = 0, \quad \pi > 0, \ \alpha > 0. \quad \textbf{(8)}$$

In the above equation, it is assumed that (*A*, *G*, *C*) is minimum-phase; that is, all system zeros z_i of (*A*, *G*, *C*) are in the open-loop left half plane. The design parameter *α* is chosen to be smaller than the minimum of $|\mathrm{Re}(z_i)|$ so that the assumption (*A*, *G*, *C*) is minimum-phase ensures that (*A* + *αI*, *G*, *C*) is also minimum-phase. Previous studies [24] on the nonlinear disturbance observer require the solution of an LMI problem, which is equivalent to an H_∞ type Riccati equation. However, their Riccati equation admits a positive definite solution matrix only if the Lipschitz constant *γ* in (3) is small. In contrast, this paper employs an LQ type Riccati equation (8), which is *γ* independent. The existence and positive definiteness of the solution matrix *Q* of the *LQ* Riccati equation (8) is always guaranteed [32] and [33] for all design parameters *π* > 0 and *α* > 0 if (*A*, *C*) is observable and (*A*, *G*) controllable.

Remark 1

Note that the above proposed robust observer is different from the linear high gain observer [34] because the proposed observer is nonlinear due to a nonlinear term $f(\hat{x})$ in (5). Furthermore, even if a linear high gain observer can work in state estimation, there is no disturbance estimation mechanism in a linear high gain observer.

The solution matrix $Q \in R^{n\times n}$ in (8) depends on its design parameter *π*, and *Q(π)* and π satisfies the following relationship.

Lemma 1 [30], [35]

If the system (*A*, *G*, *C*) is square and minimum-phase, the solution *Q(π)* of the observer Riccati equation (8), satisfies $\lim_{\pi\to\infty} \frac{Q(\pi)}{\pi} = 0.$

With the above lemma, one can further derive the following result.

Lemma 2

If the system (*A*, *G*, *C*) is square and minimum-phase, the observer feedback gain *L* in (7) satisfies

$$\frac{L}{\sqrt{\pi}} \to GW \text{ as } \pi \to \infty, \quad \textbf{(9)}$$

where $W \in R^{p\times p}$ is an unitary matrix satisfying $WW^T = I$.

Proof

Divide the Riccati equation (8) by π to obtain

$$\frac{Q}{\pi}(A+\alpha I)^T + (A+\alpha I)\frac{Q}{\pi} - \frac{Q}{\pi}\frac{C^T C}{1/\pi}\frac{Q}{\pi} + GG^T = 0.$$

It follows from Lemma 1 that the first and second terms in the above equation can be neglected as $\pi \to \infty$, and this results in $(\frac{QC^T}{\sqrt{\pi}})(\frac{CQ}{\sqrt{\pi}}) \to GG^T$.

Solving this equation yields $QC^T \to \sqrt{\pi}GW$ for some unitary matrix *W*. Finally, quoting (7) gives $L \to \sqrt{\pi}GW$. End of proof.

Remark 2

The unitary matrix W in Lemma 2 can be obtained from (9) as follows. $W \to \frac{1}{\sqrt{\pi}}(G^T G)^{-1} G^T L$ as $\pi \to \infty$.

The state estimation error $\tilde{x} = x - \hat{x}$ resulting from the above proposed nonlinear observer satisfies

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + G(f(x) - f(\hat{x}) + d)\,. \tag{10}$$

The goal of the following theorem is to prove that even for large Lipschitz constant in (3), the state estimation error $\tilde{x}$ decays to almost zero in the error dynamics (10) if the design parameter π in (8) is chosen sufficiently large.

Theorem 3

Consider the nonlinear system (1) with (*A*, *G*, *C*) square and minimum-phase. For whatever large Lipschitz constant γ in (3), the state estimation error resulting from the nonlinear observer (5) asymptotically satisfies

$$\lim_{t\to\infty} \|\tilde{x}\| \le \varepsilon, \tag{11}$$

Where ε can be an arbitrarily small positive constant if the design parameter $\pi > 0$ in the Riccati equation (8) is chosen sufficiently large.

Proof

Define a Lyapunov function $V = \tilde{x}^T Q^{-1} \tilde{x}$, where $Q > 0$ is from the observer Riccati equation (8), and x as in (10). The time change rate of *V* along the trajectory (10) satisfies

$$\dot{V} \le -2\alpha V - \|\tilde{y}\|^2 - \pi\|G^T Q^{-1}\tilde{x}\|^2 + 2(\|f(x) - f(\hat{x})\| + \|d\|)\cdot\|G^T Q^{-1}\tilde{x}\| \le -2\alpha V - \|\tilde{y}\|^2 - \pi\|G^T Q^{-1}\tilde{x}\|^2 + 2(\gamma\|\tilde{x}\| + D_0)\cdot\|G^T Q^{-1}\tilde{x}\|,$$

Where $\tilde{y} = C\tilde{x}$ and the Lipschitz condition (3) was used to obtain the second inequality. Note that the maximum of the last two terms in the last equation occurs when $\|G^T Q^{-1}\tilde{x}\| = (\gamma\|\tilde{x}\| + D_0)/\pi$, with the maximum value being

$(\gamma\|\tilde{x}\|+D_0)^2/\pi$. Hence,

$$\dot{V} \le -2\alpha V - \|\tilde{y}\|^2 + \frac{(\gamma\|\tilde{x}\|+D_0)^2}{\pi} \le -\alpha V - (\alpha V - \frac{(\gamma\|\tilde{x}\|+D_0)^2}{\pi}) \quad \textbf{(12)}$$

From the last inequality, one observes that V decays exponentially as long as $V > (\gamma\|\tilde{x}\|+D_0)^2/(\alpha\pi)$. Therefore, eventually we have $V \le (\gamma\|\tilde{x}\|+D_0)^2/(\alpha\pi)$ as $t \to \infty$. Using $V \ge \underline{\sigma}(Q^{-1})\|\tilde{x}\|^2 = \|\tilde{x}\|^2/\bar{\sigma}(Q)$, we obtain $\|\tilde{x}\| \le \sqrt{\frac{\bar{\sigma}(Q/\pi)}{\alpha}}(\gamma\|\tilde{x}\|+D_0)$ as $t \to \infty$. Re-arranging the equation gives $\lim_{t\to\infty}(1-\gamma\sqrt{\frac{\bar{\sigma}(Q/\pi)}{\alpha}})\|\tilde{x}(t)\| \le D_0\sqrt{\frac{\bar{\sigma}(Q/\pi)}{\alpha}}$. Lemma 1 ensures that given any larege Lipschitz constant γ, there always exists a sufficiently large observer design parameter π such that the number in the parenthesis in the above equation is positive. Hence, we can write $\lim_{t\to\infty}\|\tilde{x}(t)\| \le \varepsilon = \frac{D_0\sqrt{\frac{\bar{\sigma}(Q/\pi)}{\alpha}}}{1-\gamma\sqrt{\frac{\bar{\sigma}(Q/\pi)}{\alpha}}}$.

The above inequality together with Lemma 1 imply that given any bounded Lipschitz constant γ and disturbance upper bound D_0, $\|\tilde{x}\|$ will eventually become arbitrarily small as long as π is sufficiently large. End of proof.

Compared with the nonlinear state observer in [28], [29], the nonlinear observer in Theorem 3 is *robust* against unknown disturbance; it achieves arbitrarily small state estimation error in the fact of large unknown disturbance. Furthermore, even facing with arbitrarily large Lipschitz constant, the proposed observer is still stable.

In the formulation of this paper, the disturbance d enters the system through a constant vector channel G in the system equation (1). For certain nonlinear systems, for instance, the robot systems or other nonlinear systems in [29], the system has the structure that the input channel vector $G(x)$ is state dependent. In this case, we can use state transformation so that in the transformed state, the new system has a constant disturbance channel vector. For example, a nonlinear system

$$\dot{x} = Ax + G(x)d, \quad G(x) = G_1 n_1(x) + G_2 n_2(x),$$

Where G_1 and G_2 are constant vectors, and $n_1(x)$ and $n_2(x)$ are nonlinear functions of x. If we defines extended state $z = [x, x_{n+1}]$, where $x_{n+1} = d$, then the above system can be written as

$$\dot{x} = Ax + G_1 n_1(x) x_{n+1} + G_2 n_2(x) x_{n+1}, \quad \textbf{(13)}$$

$$\dot{x}_{n+1} = d_1, \quad \text{where } d_1 = \dot{d}. \quad \textbf{(14)}$$

The above new system can be viewed as a nonlinear system with nonlinearity $n_1(x)x_{n+1}$ and $n_2(x)x_{n+1}$ and an disturbance d_1 with constant disturbance channel vector. We can then follow the procedure in the next section to design state observer and disturbance observer.

The state estimation error dynamics resulting from the proposed nonlinear observer can be written as, via the result of Lemma 2,

$$\dot{\tilde{x}} = A\tilde{x} + G(f(x) - f(\hat{x}) + d) - LC\tilde{x} = A\tilde{x} + Gv, \text{ as } \pi \to \infty, \quad (15)$$

$$v = f(x) - f(\hat{x}) + d - \sqrt{\pi}WC\tilde{x} = f(x) - f(\hat{x}) + d - \hat{d}. \quad (16)$$

We can view the above error dynamics (15) as a linear system subject to an input excitation *v*. According to Theorem 3, the system state $\tilde{x}$ becomes arbitrarily small if the design parameter *π* of the nonlinear observer is chosen sufficiently large. One question stemming from equation (15) is that since the state $\tilde{x}$ is small, can one conclude that its input excitation *v* is also small ? This is the bounded state bounded input (BSBI) property. A counter example of BSBI property is the sliding mode control applied to a linear system, $\dot{x} = Ax + B(u+d),\ u = -\rho \cdot sign(s),$ where the system is subject to an unknown disturbance *d*, *ρ* is the switching control gain, and *s* is the sliding variable. If *s* and *ρ* are properly chosen, the switching control *u* can drive the system state *x* to zero. In this case, the state *x* is small, but the input excitation *u* + *d* is not necessarily small.

To answer the question of BSBI property, one quotes a well-known lemma in the study of Adaptive Control. Define the truncated norm of a time signal *v* as $\|v\|_t = \sup_{T\le\tau\le t}\|v(\tau)\|$. A time signal *v* is *regular* if $\|\dot{v}\| \le M_1\|v\|_t + M_0$ for two positive constants M_1 and M_0 for *t* > *T*. Due to (11) in Theorem 3, there exists a finite time *T* such that

$$\|\tilde{x}(t)\| \le 1.5\varepsilon \text{ for all } t > T. \quad (17)$$

In the sequel, this *T* will be used in the definition of $\|v\|_t = \sup_{T\le\tau\le t}\|v(\tau)\|$. Hence, it follows from (17) that

$$\|\tilde{x}(t)\| \le 1.5\varepsilon \text{ as } t \to \infty. \quad (18)$$

Note that the disturbance observer design problem for nonlinear systems is more difficult than the state observer design problem for nonlinear systems. In the disturbance observer problem, one must first design a robust nonlinear observer that ensure small estimation error in front of unknown disturbances. Then, we need to prove for the state estimation error dynamics that small state estimation error implies small input excitation. After establishing this property we can prove that the disturbance estimation error is small. The latter job is non-trivial, and is dealt with by Lemma 4 and Theorem 5 below.

Lemma 4 [36]

Given a linear system $\dot{\tilde{x}} = A\tilde{x} + Gv$, where (*A*, *G*) is controllable, if *v* is regular, then

$$\|v\|_t \le K\|\tilde{x}\|_t, \quad (19)$$

For some positive constant *K*

Lemma 4 states that as long as the input is smooth in the sense of being regular, we can use (19) to conclude BSBI property. With Lemma 4, one is now in a position to prove the main result of this section that $\hat{d}$ in (6) is a close estimate of the unknown disturbance *d*.

Theorem 5

Consider the nonlinear system (1) subject to the unknown disturbance *d*, and the corresponding nonlinear observer (5) and the disturbance estimate $\hat{d}$ in (6). If functions $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial x}Ax$, $\frac{\partial f}{\partial x}Gf(x)$, are all Lipschitz, and the control input *u* is bounded, the disturbance estimate $\hat{d}$ in (6) is arbitrarily close to the true disturbance *d* if the observer design parameter π in (8) is sufficiently large.

Proof

Define a function

$$M(x,t)=\frac{\partial f}{\partial x}Ax+\frac{\partial f}{\partial x}Bu+\frac{\partial f}{\partial x}G(f(x)+d).$$

From hypothesis of this theorem, *M* is Lipschitz in *x*. According to the definition of $v=f(x)-f(\hat{x})+d-\sqrt{\pi}WC\tilde{x}$ in (16), the time derivative of *v* can be expressed as $\dot{v}=M(x,t)-M(\hat{x},t)+\dot{d}-\sqrt{\pi}WC(A\tilde{x}+Gv)$.

Finally, quoting the Lipschitzness of *M*(*x*,*t*), boundedness of $\dot{d}$, and Theorem 3, one concludes from the above equation that *v* is regular. Since *v* in (15) is regular, we deduces from Lemma 4 that $\|v(t)\|\leq\|v\|_t\leq K\|\tilde{x}\|_t\leq 1.5K\varepsilon$ as $t\to\infty$, where the last inequality follows from (18). Define the disturbance estimation error $\tilde{d}=d-\hat{d}$, and re-arrange (16) to yield

$\|\tilde{d}\|=\|d-\hat{d}\|=\|v-f(x)+f(\tilde{x})\|\leq\|v\|+\gamma\|\tilde{x}\|\leq 1.5K\varepsilon+\gamma\varepsilon$, where we have used the Lipschitz condition (3) to obtain the first inequality. Since ε approaches zero as the design parameter π approaches infinity, the last inequality shows that the disturbance estimation error d can be made arbitrarily small. End of proof.

The above theorem shows that the disturbance estimate d is arbitrarily close to the true disturbance *d* if the observer design parameter π is sufficiently large. In real applications, we can use computer simulations to test just how large the design parameter π should be so that a desired disturbance estimation error is obtained.

3. EXTENSION TO MORE GENERAL SYSTEMS

In previous sections, the unknown disturbance and the nonlinear function enter the state equation through the same channel. The purpose of this section is to construct a disturbance observer for systems whose unknown disturbance and nonlinear function enter the state equation through *different* channels. Consider a nonlinear system subject to an unknown disturbance,

$$\begin{aligned}\dot{x}&=Ax+Bu+G_1f(x)+G_2d,\\ y&=Cx.\end{aligned}\tag{20}$$

Where all symbols are as defined in (1), and $G_1\neq G_2$. Further, it is assumed that (*A*, *C*) is observable, and (*A*, [G_1, G_2]) controllable, $f(x)\in R^{q1}, d\in R^{q2}$, and $y\in R^p$.

Constructing a disturbance observer for the above system is more difficult than for the system (1) in the previous section. However, it will be shown that if the number of output sensors is sufficient in the sense that

$$dim(y) \geq dim(f(x)) + dim(d) \Leftrightarrow p \geq q1 + q2 \ , \quad \textbf{(21)}$$

Where *dim* stands for dimension, then design of a nonlinear disturbance observer is achievable. For simplicity, it will be assumed in the sequel that $dim(y) = dim(f(x)) + dim(d)$, $rank(C) = rank([G_1, G_2])$.

The case $dim(y) > dim(f(x)) + dim(d)$ can be treated similarly by artificially expanding the dimension of *d* or *f*(*x*). Note that this condition (21) is similar to the assumption adopted in other reference [24].

A nonlinear disturbance observer for the system (20) is proposed as follows,

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\tilde{x}) + G_1 f(\hat{x}), \quad \textbf{(22)}$$

$$\hat{d} = [0_{q2\times q1}, \ I_{q2\times q2}]\sqrt{\pi}W(y - C\hat{x}), \quad \textbf{(23)}$$

Where *W* is an unitary matrix to be specified, and the observer feedback gain $L \in R^{n\times p}$ is given by

$$\begin{aligned} & L = QC^T, \\ & Q(A+\alpha I)^T + (A+\alpha I)Q - QC^TCQ + \pi[G_1, G_2][G_1, G_2]^T = 0, \ \pi > 0, \ \alpha > 0. \end{aligned} \quad \textbf{(24)}$$

Note that the above Riccati equation is different from that in the previous section. The existence of a positive definite solution matrix $Q \in R^{n\times n}$ is guaranteed [32] and [33] when (*A*, *C*) is observable, and (*A*, [G_1 , G_2]) controllable.

The state estimation error $\tilde{x} = x - \hat{x}$ resulting from the proposed robust nonlinear observer (22) obeys the dynamics $\dot{\tilde{x}} = (A - LC)\tilde{x} + [G_1, \ G_2]\begin{bmatrix} f(x) - f(\hat{x}) \\ d \end{bmatrix}$ where the last term satisfies an upper bound $\left\| \begin{bmatrix} f(x) - f(\hat{x}) \\ d \end{bmatrix} \right\| \leq \| f(x) - f(\hat{x}) \| + \|d\| \leq \gamma \|\tilde{x}\| + D_0$ in which we have used (3) and (2) to obtain the second inequality.

The following theorem shows that the above robust nonlinear observer (22) guarantees that the estimation error $x - \hat{x}$ decays to almost zero even in the face of large disturbance *d*.

Theorem 6

Consider the nonlinear system (20) subject to a bounded disturbance *d*, and the assumption that (*A*, [G_1 , G_2], *C*) is square and minimum-phase. For whatever large Lipschitz constant *γ*, and whatever large disturbance *d*, the robust nonlinear observer (22) guarantees that the state estimation error $\tilde{x} = x - \hat{x}$ will eventually approach a small residual set containing the origin, with the size of the residual set approaching zero as the design parameter *π* in (24) approaches infinity.

Proof

The proof is similar to that of Theorem 3, and is omitted.

The next theorem proves that the disturbance estimate $\hat{d}$ in (23) is arbitrarily close to the true disturbance d if the design parameter π in the Riccati equation (24) is chosen sufficiently large.

Theorem 7

Consider the nonlinear system (20) subject to the unknown disturbance d, and the corresponding nonlinear observer (22). If functions $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial x}Ax$, $\frac{\partial f}{\partial x}Gf(x)$, are all Lipschitz, and the control input u is bounded, the disturbance estimate $\hat{d}$ in (23) can be arbitrarily close to the true disturbance d if the observer design parameter π in (24) is sufficiently large.

Proof

The proof is similar to that of Theorem 5, and is omitted.

To demonstrate the effectiveness of the proposed nonlinear disturbance observer, we present a numerical simulation example.

Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix},\ B = G_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ G_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ C^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Consider a single-link with flexible joint robot [37], whose governing equation is as in (20) with the control $u(t)$ = 0, the disturbance $d(t) = 2$, and nonlinearity $f(x) = -10\sin x_1$. Note that the nonlinearity $f(x)$ is globally Lipschitz, and the large Lipschitz constant $\gamma = 10$ implies a long or heavy robot arm. The initial condition are $x^T(0) = [1,\ 1,\ 1,\ 1]$ for the robot, and $\hat{x}^T(0) = [-1,\ -1,\ -1,\ -1]$ for the observer (22). The modified Riccati equation (24) has design parameters $\alpha = 1$, and $\pi(t)$ scheduled continuously from 1 to 10^{12} within 10 seconds according to the formula $\pi(t) = 10^{12t/10}$, $t \in [0, 10]$ and $\pi(t) = 10^{12}$, $t > 10$. Note that the design parameter π is time-varying only for the initial transient 10 seconds. After $t > 10$ seconds, the design parameter π becomes constant. Hence, the analysis in the previous sections applies Theorems 3 and 5 holds for $t > 10$. At the steady state, $\pi = 10^{12}$, the corresponding observer feedback gain is found to be

$$L = \begin{bmatrix} 7.0711e-004 & 1.4162e+003 \\ 1.0004 & 1.0014e+006 \\ 1.4152e+003 & 7.0711e-004 \\ 1.0000e+006 & 1.0004 \end{bmatrix}$$

Figure 1 shows the disturbance d and the proposed disturbance estimate $\hat{d}$. It is seen that the $\hat{d}$ converges to d, as predicted by Theorem 5, in about 4 seconds. Figure 2 shows the state estimation error $\|\tilde{x}(t)\|$ versus time. The estimation error decays to almost zero $(\|\tilde{x}(t)\| \approx 10^{-3})$, confirming that the proposed robust nonlinear observer design is successful in the face of disturbances and large-Lipschitz-constant nonlinearity. Note that if we use $\pi = 10^{12}$ right from the beginning of observation, there will be a substantial peaking phenomenon. The proposed stepwise scheduling of $\pi(t)$ has successfully avoided the peaking phenomenon.

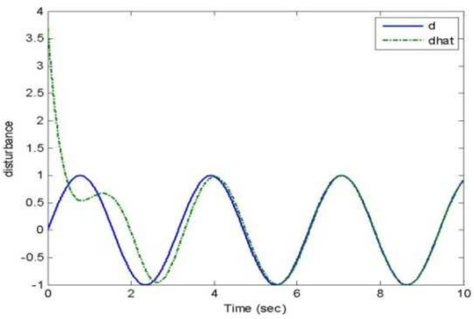

Figure 1: Disturbance and Its Estimate

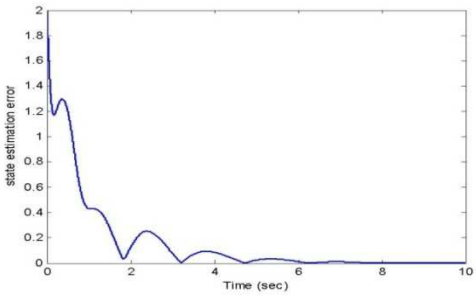

Figure 2: Norm of State Estimation Norm

4. CONCLUSIONS

Previous studies on the disturbance observer are mostly linear systems. This paper proposes a nonlinear disturbance observer for a class of nonlinear systems. Arbitrarily close estimate of the unknown disturbance can be obtained if the disturbance is bounded with bounded first-order time derivative, and the system nonlinearity satisfies a Lipschitz condition. The contribution of this paper is that it allows the Lipschitz constant to be arbitrarily large, and it imposes no constraint on the system relative degree.

5. ACKNOWLEDGEMENTS

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6. REFERENCES

1. D. M. Trujillo and H. R. Busby (1997). *Practical Inverse Analysis in Engineering*, CRC Press.
2. J. C. Santamarina and D. Fratta (2005). *Discrete Signals and Inverse Problems: An Introduction for Engineers and Scientists*, John Wiley and Sons, Ltd.
3. A. Radke and Z. Gao (2006). "A survey of state and disturbance observers for practitioners," *Proc. of the American Control Conference ACC/IEEE*, pp. 5183 – 5188.
4. M. Nakao, K. Ohnishi, and K. Miyachi (1987). "A robust decentralized joint control based on interference estimation," *Proc. IEEE Int. Conf. Robotics and Automation*, pp. 326-331.

5. K. Ohnishi, M. Shibata, and T. Murakami (1996). "Motion control for advanced mechatronics," *IEEE/ASME Trans. Mechatronics*, vol. 1, no. 1, pp. 56 -67.

6. T. Umeno, T. Kaneko, and Y. Hori (1993). "Robust servo system design with two degrees of freedom and its application to novel motion control of robot manipulators," *IEEE Trans. Ind. Electron.*, vol. 40, pp. 473 – 485.

7. S. Komada, N. Machii, and T. Hori (2000). "Control of redundant manipulators considering order of disturbance observer," *IEEE Trans. Ind. Electron.*, vol. 47, no. 2, pp. 413 - 420.

8. Y. Choi, K. Yang, W. K. Chung, H. R. Kim, and I. H. Suh (2003). "On the robustness and performance of disturbance observers for second order systems," *IEEE Trans. Automatic Control*, vol. 48, pp. 315-320.

9. H. Shim and N. H. Jo (2009). "An almost necessary and sufficient condition for robust stability of closed-loop systems with disturbance observer," *Automatica*, vol. 45, pp. 296 - 299.

10. A. Tesfaye, H. S. Lee, and M. Tomizuka (2000). "A sensitivity optimization approach to design of a disturbance observer in digital motion control systems," *IEEE/ASME Trans. Mechatronics*, vol. 5, no. 1, pp. 32 - 38.

11. B. K. Kim and W. K. Chung (2003). "Advanced disturbance observer design for mechanical positioning systems," *IEEE Trans. Ind. Electron.*, vol. 50, no. 6, pp. 1207 - 1216.

12. H. Wang and S. Daley (1996). "Actuator fault diagnosis: an adaptive observer-based technique", *IEEE Trans. Automatic Control*, vol. 41, pp. 1073 - 1078.

13. D. Haessig and B. Friedland (1998). "Separate-bias estimation with reduced-order Kalman filters", *IEEE Trans. Automatic Control*, vol. 44, pp. 983 - 987.

14. J. L. Chang (2006). "Applying discrete-time proportional integral observers for state and disturbance estimation", *IEEE Trans. Automatic Control*, vol. 51, no. 5, pp. 814 - 818.

15. J. Q. Han (1995). "A class of extended state observers for uncertain systems," *Control and Decision*, vol. 10, no. 1, pp. 85 - 88, in Chinese.

16. Z. Gao, Y. Huang, and J. Han (2001). "An alternative paradigm for control system design," *Proc. IEEE Conf. Decision and Control*, pp. 4578 - 4585.

17. Z. Gao (2006). "Active disturbance rejection control: a paradigm shift in feedback control," *Proc. of the American Control Conference ACC/IEEE*, pp. 2399- 2405.

18. C. Johnson (1971). "Accomodation of external disturbances in linear regulator and servomechanism problems", *IEEE Trans. Automatic Control*, vol. 16, no. 6, pp. 635 - 644.

19. G. Hostetter and J. S. Meditch (1973). "Observing systems with unmeasurable inputs", *IEEE Trans. Automatic Control*, vol. 18, pp. 608 - 609.

20. G. Hostetter and J. S. Meditch (1973). "On the generalization of observers to systems with unmeasurable, unknown inputs ", *Automatica*, vol. 9, pp. 721- 724.

21. J. A. Profeta, W. G. Vogt, and M. H. Mickle (1990). "Disturbance estimation and compensation in linear systems," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 26, pp. 225 - 231.

22. M. Corless and J. Tu (1998). “State and input estimation for a class of uncertain systems”, *Automatica*, vol. 34, pp. 757 - 764.

23. Y. Xiong and M. Saif (2003). “Unknown disturbance inputs estimation based on a state functional observer design”, Automatica, vol. 39, pp. 1389 - 1398.

24. Q. P. Ha and H. Trinh (2004). “State and input simultaneous estimation for a class of nonlinear systems,” *Automatica*, vol. 40, pp. 1779-1785.

25. W. H. Chen, D. J. Ballance, P. J. Gawthrop, and J. O’Reilly (2000). “A nonlinear disturbance observer for robotic manipulators”, *IEEE Trans. Industrial Electronics*, vol. 47, no. 4, pp. 932 - 938.

26. N. Amin and H. Reza (2009). “Lyapunov-based nonlinear disturbance observer for serial n-link robot maniplulators,” *J. Intell Robot Syst.*, vol. 55, pp. 135-153.

27. A. Mohammadi, M. Tavakoli, H. J. Marquez, and F. Hashemzadeh (2013). “Nonlinear disturbance observer design for robotic maniplulators,” *Control Engineering Practice*, vol. 21, pp. 253-267.

28. R. Rajamani and Y. M. Cho (1998). “Existence and design of observers for nonlinear systems: relation to distance to unobservability,” *Int. J. Control*, vol. 69, no. 5, pp. 717-731.

29. J. P. Gauthier, H. Hammouri, and S. Othman (1992). “A simple observer for nonlinear systems, applications to bioreactors,” *IEEE Trans. Automat. Control*, vol. 37, pp. 875-880.

30. J. C. Doyle and G. Stein (1979). “Robustness with observers,” *IEEE Trans. Automat. Control*, vol. 24, pp. 607-611.

31. G. Stein, and M. Athans (1987). “The LQG/LTR procedure for multivariable feedback control design,” *IEEE Trans. Automat. Control*, vol. 32, pp. 105-113.

32. A. Saberi, P. Sannuti, and B. M. Chen (1995). H_2 *Optimal Control*, Theorem 4.1.2 in Chapter 4, New York, Prentice-Hall

33. F. L. Lewis (1992). *Applied Optimal Control And Estimation*, Theorem 2 in Chapter 3, Englewood Cliffs, New Jersey, Prentice-Hall.

34. H. K. Khalil (2000). *Nonlinear Systems*, Upper Saddle River, New Jersey, Prenticce-Hall.

35. H. Kwakernaak and R. Sivan (1972). *Linear Optimal Control Systems*, Theorem3.14 in Chapter 3, New York, Wiley.

36. K.S. Narendra and A. M. Annaswamy (1989). *Stable Adaptive Systems*, page 80, Englewood Cliff, New Jersey, Prentice-Hall.

37. R. Marino and P. Tomei (1995). *Nonlinear Control Design*, page 27, New York, Prentice-Hall.